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The Largest Singular Value of $e^{X} A_{0} e^{-X}$ is Convex on Convex Sets of Commuting Matrices

R. Sezginer
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NEW YORK UNIVERSITY

Department of Computer Science
Courant Institute of Mathematical Sciences
251 MERCER STREET, NEW YORK, N.Y. 10012

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The Largest Singular Value of $e^X A_0 e^{-X}$ is Convex on Convex Sets of Commuting Matrices

Renan Sezer Sezginer and Michael L. Overton*

Courant Institute of Mathematical Sciences
New York University
251 Mercer St., New York, NY 10012

Abstract

A short and direct proof is given for the convexity of the largest singular value of $e^X A_0 e^{-X}$ on any convex set of commuting matrices in $R^{n \times n}$, where $A_0 \in R^{n \times n}$ is fixed. This statement does not hold if X is permitted to be a general square matrix. A counterexample is supplied for noncommuting matrices.

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Doyle's structured singular value analysis [1] is a well known tool in robust control. One technique (see [1,2]) used to bound the structured singular value is to minimize the largest singular value, i.e. the two-norm, of the matrix DA_0D^{-1}, where A_0 is fixed and D is a variable positive diagonal matrix. Doyle and Chu showed that the largest singular value of this matrix is a convex function of X, where $D = e^X$. Hence, the optimization problem is tractable. Their proof consists of two steps, neither of which is readily accessible in the open literature. The first step, a characterization of convexity, may be found in [3] and the second step may be found in [4].

In this note we give a new, short, and direct proof of the convexity property. We show that the theorem applies to any convex, commuting set of matrices X. We also show that the result does not hold if X is permitted to be a general square matrix.

Theorem: Let Ω be a convex subset of $R^{n\times n}$, such that $X, Y \in \Omega \Longrightarrow XY = YX$. For any $A_0 \in R^{n\times n}$, the l_2-norm of the matrix valued function $A : \Omega \mapsto R^{n\times n}$ defined by

$$A(X) = e^X\, A_0\, e^{-X} \tag{1}$$

is convex on Ω.

Proof: Since $\|A(X)\|$ is a continuous function, it suffices to show that

$$\frac{1}{2}\|A(X)\| + \frac{1}{2}\|A(Y)\| \geq \|A(Z)\|\,, \quad \text{where } Z = (X+Y)/2,\ \forall X, Y \in \Omega, \tag{2}$$

in order to prove convexity [5, p. 101]. In the following argument, S^{n-1} denotes the unit sphere $\{\xi \in R^n|\ \ \|\xi\| = 1\}$. We define the matrix

$$E = e^{(Y-X)/2}\,, \tag{3}$$

and the right and left singular vectors, v and u, of $A(Z)$ corresponding to its largest singular value:

$$A(Z)v = \|A(Z)\|\, u \tag{4}$$

$$A^T(Z)u = \|A(Z)\|\, v \tag{5}$$

Using the triangle inequality and the hypothesis $XY = YX$, we have:

$$\frac{1}{2}\|A(X)\| + \frac{1}{2}\|A(Y)\| \quad \geq$$

$$\geq \frac{1}{2}\|A(X) + A(Y)\| \tag{6}$$

$$= \frac{1}{2}\left\|E^{-1}A(Z)E + EA(Z)E^{-1}\right\| \tag{7}$$

$$= \frac{1}{2} \sup_{\zeta,\xi \in S^{n-1}} \zeta^T \left(E^{-1}A(Z)E + EA(Z)E^{-1}\right)\xi \tag{8}$$

$$\geq \frac{1}{2} \frac{u^T E}{\sqrt{u^T E^2 u}} \left(E^{-1}A(Z)E + EA(Z)E^{-1}\right) \frac{Ev}{\sqrt{v^T E^2 v}} \tag{9}$$

$$= \frac{1}{2} \frac{v^T E^2 v \; + \; u^T E^2 u}{\sqrt{u^T E^2 u}\,\sqrt{v^T E^2 v}} \quad \|A(Z)\| \tag{10}$$

$$= \frac{1}{2} \left(\sqrt{\frac{v^T E^2 v}{u^T E^2 u}} + \sqrt{\frac{u^T E^2 u}{v^T E^2 v}}\right) \quad \|A(Z)\| \tag{11}$$

$$\geq \|A(Z)\| \tag{12}$$

The last inequality follows from the fact that

$$\frac{1}{2}\left(\theta + \frac{1}{\theta}\right) \geq 1\,, \quad \forall \theta > 0\,. \tag{13}$$

An example of a convex, commuting subset of $R^{n\times n}$, is:

$$\Psi_M = \left\{M \operatorname{diag}(x) M^{-1} \,\middle|\; x \in R^n\right\} \tag{14}$$

where $M \in R^{n\times n}$, $\det(M) \neq 0$, and $\operatorname{diag}(x)$ denotes the $n \times n$ diagonal matrix which satisfies $(\operatorname{diag}(x))_{ii} = x_i$. In particular, Ψ_I is the set of diagonal matrices.

It has generally been thought that Doyle and Chu's proof applies to general square matrices. However, this is not true, as the following counterexample shows:

$$A_0 = \begin{bmatrix} -0.88 & 1.00 \\ 1.00 & 0.88 \end{bmatrix}$$

$$X = \begin{bmatrix} 1.53 & 0.22 \\ 0.22 & 4.40 \end{bmatrix}$$

$$Y = \begin{bmatrix} 1.28 & 2.31 \\ 2.31 & 5.32 \end{bmatrix}$$

$$\frac{\|A(X)\|}{2} + \frac{\|A(Y)\|}{2} \simeq 9.1361 \qquad (15)$$

$$\left\| A\left(\frac{X+Y}{2}\right)\right\| \simeq 20.8545$$

Finally, we note that although $\|A(X)\|$ is convex on Ω, it is not differentiable everywhere, since the singular values of a matrix are generally not differentiable at points where they coalesce to multiple values. This is well known and treated in [1]; another relevant reference is [6]. See [7] for a recent contribution showing how to obtain an algorithm which is quadratically convergent even to nondifferentiable solutions. The algorithm is developed for the related problem of minimizing the maximum eigenvalue of a linear combination of symmetric matrices; we are currently extending this to minimize (1).

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